Micro-Foundation of ARCH Model

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Many macroeconomic study argued macroeconomic models should be aggregated by micro processes models ("micro-foundation") and many micro-founded macroeconomic models were built. On the other hand, there are many models for price variation of a risk asset, which is macro phenomena, however, there are few studies for micro-foundation of such models. In this study we tried micro-foundation of an ARCH model using intelligence of artificial market simulation studies. That is we tried to clarify which micro processes determine each coefficient of an ARCH model. Then, we showed that the dispersion of investors' estimated prices is larger or the orders by the buy-sell imbalance taking liquidity are more, the volatility is larger. And we showed that the ration of the normal investors taking liquidity to the noise traders providing liquidity is higher or the measure of risk aversion of the normal investors is lower, the magnitude of volatility clustering is larger.

1. Introduction

Many macroeconomic study argued macroeconomic models should be aggregated by micro processes models ("micro-foundation")*1, and many micro-founded macroeconomic models were built. On the other hand, there are many models for price variation of a risk asset, which is macro phenomena, such as ARCH model [Engle 82] and GARCH model [Bollerslev 86], however, there are few studies for micro-foundation of such models than macroeconomic models.

On the other hand, there are many previous study using artificial market simulation models, a kind of multi-agent models, which simulate macro processes such as investors and order matching on a computer. Artificial market studies observe macro phenomena such as price variations as a result of the modeled macro processes*2.

Not only academies but also financial regulators and stock exchanges are recently interested in multi-agent simulations such artificial market models to investigate regulations and rules of financial markets. Indeed, the Science article by Battiston et al. [Battiston 16] described that 'since the 2008 crisis, there has been increasing interest in using ideas from complexity theory (using network models, multi-agent models, and so on) to make sense of economic and financial markets'. Actually, some artificial market studies recently contributed to discussion what financial regulations and rules should be*3.

Artificial market models only modelize the micro processes and observe macro phenomena, therefore, artificial market models are fully micro-founded models. So, artificial market models have been gaining intelligence micro-macro interaction mechanisms such as what micro processes amplify price variations.

Then in this study we try micro-foundation of the ARCH(1) model [Engle 82] using intelligence of artificial market simulation studies. That is we try to clarify which micro processes determine each coefficient of the ARCH(1) model.

2. Model

An ARCH(1) model [Engle 82] with an average return is zero is finally derived,

$$\begin{aligned} r_t &= \sigma_t \epsilon_t \\ \sigma_t^2 &= a_0 + a_1 r_{t-1}^2. \end{aligned} \tag{1}$$

Where r_t is a log return, ϵ_t is a stochastic variable obeying a standard normal distribution, and a_0, a_1 are constants. a_0 indicates a magnitude of a volatility (a degree of variation of a market price series) and a_1 indicates a magnitude of a volatility clustering (a large volatility tend to maintain a large for a while) [Mandelbrot 63]. When $a_1 = 0$, the model indicates the case without a volatility clustering. Therefore, micro processes determining a_0 are cause of an ever-present volatility, and micro processes determining a_1 are cause of an additional volatility by the volatility clustering.

Next, we build a macro model aggregated by micro process discovered by the artificial market studies, and we compare the model with Eq.(1).

There are many artificial market models adopting a pricing model which determines a price variation by buy-sell order imbalance*4. Even though they adopt such simple pricing model, it is well known that they can replicate statistics of price variations in real financial markets. For example, [Palmer 94] defined $r_t = \eta(A_t^b - A_t^s)$ where r_t, A_t^b and A_t^s are a return from time $t-1$ to t, an amount of buy and sell orders respectively. η is some constant, and [Palmer 94] did not show that what micro processes in real markets determine η. [Mizuta 12] tried to clarify which micro processes determine η and obtained,

$$r_t = \rho \frac{A_t^b - A_t^s}{A_t^b + A_t^s}. \tag{2}$$

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*1 [Lucas 76] is very famous example.

*2 [LeBaron 06, Chen 12, Cristelli 14, Mizuta 16a] are excellent reviews.

*3 for example, tick size [Mizuta 13], speed of order matching systems on financial exchanges [Mizuta 15a], short selling regulation [Mizuta 16b] and usage rate of dark pools [Mizuta 15b].

*4 For examples, [Arthur 91, Palmer 94, Arthur 97, Lux 99].

Where ρ is proportional to a dispersion of investors' estimated prices. In short, the dispersion is larger, the volatility is larger. We explain details how they obtain Eq.(2) in the Appendix.

At first, we discuss the case only with noise traders. We assume,

$$\begin{aligned} A_t^b &= \tfrac{1}{2}S + \tfrac{1}{2}kS\epsilon_t, \\ A_t^s &= \tfrac{1}{2}S - \tfrac{1}{2}kS\epsilon_t. \end{aligned} \tag{3}$$

Where S and k are constants. The first term describes orders that noise traders always make and the second term describes the buy-sell order imbalance. k describes the amount of order by the imbalance per the amount of ever-present orders. Therefore, k means how much liquidity the orders by the imbalance take, and k is larger, the liquidity is lower.

To substitute Eq.(2) for Eq.(3) we obtain,

$$r_t = \rho k \epsilon_t. \tag{4}$$

To compare Eq.(1) with this we obtain,

$$\sigma_t^2 = \rho^2 k^2 = a_0. \tag{5}$$

This shows the case with $a_1 = 0$. As I mentioned micro processes determining a_0 are cause of an ever-present volatility, and micro processes determining a_1 are cause of an additional volatility by the volatility clustering. Therefore, the noise traders only generate the ever-present volatility, and do no generate the volatility clustering. The degree of volatility is determined by the dispersion of investors' estimated prices(ρ) and the orders by the imbalance taking liquidity(k).

For examples, an asset having high dispersion of estimated prices such as a stock has larger volatility than an asset having low dispersion such as a high credit rating bond. And a stock having low liquidity due to be few waiting orders has larger volatility than a stock having high liquidity provided by many waiting orders.

Next, we discuss the case not only with the noise traders but also with the normal investors who determine amount of orders using a utility function. The normal investors correspond to fundamental and/or technical strategy investors in real financial markets. Normal investors determine optimal holdings of shares on their own strategies, and make their holdings by taking liquidity provided by the noise traders.

Here, we assume,

$$\begin{aligned} A_t^b &= \tfrac{1}{2}S + \tfrac{1}{2}kS(1 + lU_t)\epsilon_t, \\ A_t^s &= \tfrac{1}{2}S - \tfrac{1}{2}kS(1 + lU_t)\epsilon_t. \end{aligned} \tag{6}$$

lU_t is a term added due to the normal investors where l is a ratio of existing the normal investors and U_t is a utility function determining amount of orders. l means a ratio of investors taking liquidity to traders providing liquidity, and l is larger, the liquidity is lower. Here, we assume a constant absolute risk aversion (CARA) model*5 as U_t. Therefore we obtain,

$$U_t \propto \frac{r_t^e}{\alpha} \tag{7}$$

Where r_t^e is a normal investors' estimated prices, α is a measure of risk aversion. Usually, normal investors estimate larger return, the price varied larger. Therefore, we assume that r_t^e is in proportion to r_{t-1}^2 and,

$$U_t = \frac{r_{t-1}^2}{\alpha}. \tag{8}$$

To substitute Eq.(2) for Eq.(6) and Eq.(8) we obtain,

$$r_t = \rho k \big(1 + \frac{l}{\alpha} r_{t-1}^2\big)\epsilon_t. \tag{9}$$

Furthermore we assume $r_{t-1}^4 \ll 1$. Comparing Eq.(1) with this we obtain,

$$\sigma_t^2 = \rho^2 k^2 + 2\rho^2 k^2 \frac{l}{\alpha} r_{t-1}^2, \tag{10}$$

and,

$$\begin{aligned} a_0 &= \rho^2 k^2, \\ a_1 &= 2\rho^2 k^2 \tfrac{l}{\alpha}. \end{aligned} \tag{11}$$

a_0 is exactly same with the case without the normal investors. As we mentioned a_1 indicates a magnitude of the volatility clustering, and the magnitude of the volatility clustering are determined not only by ρ and k same as the ever-present volatility(a_0) but also by the ratio of existing the normal investors(l) and the measure of risk aversion of the normal investors(α).

For examples, the ration of the normal investors taking liquidity to the noise traders providing liquidity is higher, the magnitude of volatility clustering is larger. And, the measure of risk aversion of the normal investors is lower, in short investors make risk-taking trades more, the magnitude of volatility clustering is larger. On the other hand, the normal investors do not affect the ever-present volatility.

3. Conclusion and Future Work

In this study we tried micro-foundation of the ARCH(1) model [Engle 82] using intelligence of artificial market simulation studies. That is we tried to clarify which micro processes determine each coefficient of the ARCH(1) model. Then we obtained,

$$\sigma_t^2 = \rho^2 k^2 + 2\rho^2 k^2 \frac{l}{\alpha} r_{t-1}^2. \tag{12}$$

The dispersion of investors' estimated prices(ρ) is larger or the orders by the buy-sell imbalance taking liquidity(k) is larger, the volatility is larger. The ration of the normal investors taking liquidity to the noise traders providing liquidity(l) is higher or the measure of risk aversion of the normal investors(α) is lower, the magnitude of volatility clustering is larger.

There are two future works. One is an empirical study validating our model. Another one is more detail discussion of our assumptions that are too strong assumptions for real financial markets.

*5 [Pratt 64, Arrow 65] proposed CARA model. [Izumi 96, Arthur 97, Izumi 99, Yagi 10, Chiarella 09, Gsell 09] and so on used CARA model for investors to determine amount of orders. [Mizuta 12] showed that CARA model play an important role to replicate the volatility clustering.

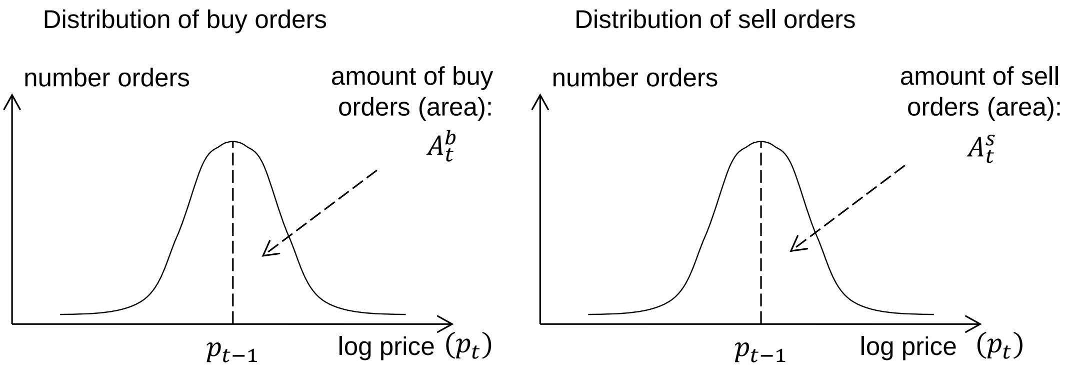

Fig. 1: Distributions of buy and sell orders.

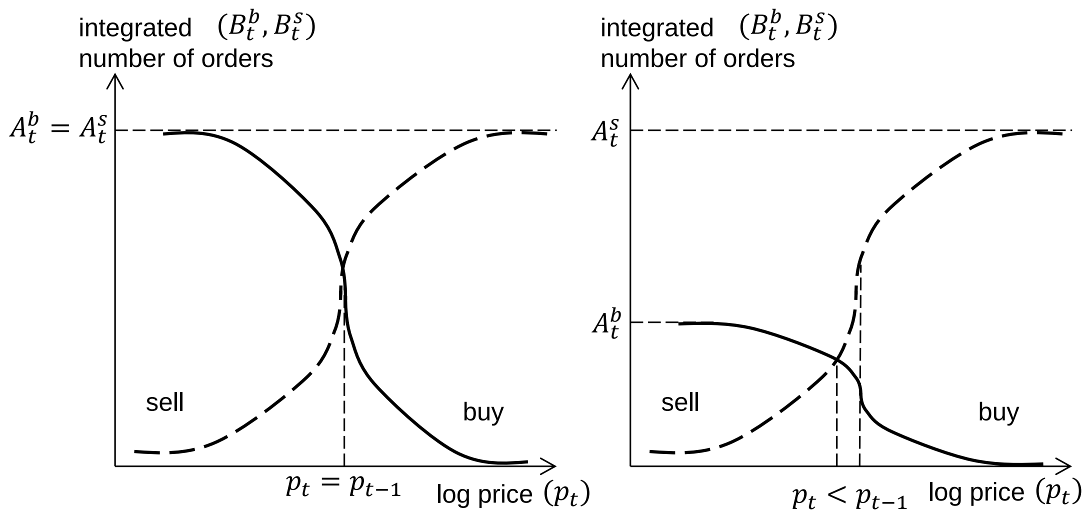

Fig. 2: Determining a trade price by the call market. Note that the horizontal axis shows prices and the vertical axis shows cumulative amount of orders.

Appendix

Here, we introduce the derivation of Eq.(2) by [Mizuta 12]. They assumed there are enough buying investors and selling investors. They assumed that investors' buy and sell order prices(log prices) p_t at time t are a normal distribution which average is p_{t-1} and standard deviation is $\acute{\rho}$ as Fig.1. The dispersion of order prices ($\acute{\rho}$) is generated by a dispersion of investors' estimated prices. The distributions of buy and sell orders are same shape, however, amounts of orders that correspond to the areas of distributions are not equal. They defined the areas of buy and sell order distributions as A_t^b and A_t^s. They employed a call auction as a pricing mechanism in which a trade price (market price) is determined at a balance of supply and demand. As Fig.2 shows, the market price is determined at an intersection of the demand curve made by the buy orders distribution and the supply curve made by the sell orders distribution*6. The supply curve $B_t^s(p_t)$ is a cumulative amount of orders less than some price p_t. Therefore $B_t^s(p_t)$ is integrated sell order distribution from the cheapest price to p_t, and then they obtained,

$$B_t^s(p_t) = \frac{A_t^s}{2}\left[1 + \mathrm{erf}\left(\frac{p_t - p_{t-1}}{\sqrt{2\acute{\rho}^2}}\right)\right]. \tag{13}$$

Where erf is the error function and defined as,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt. \tag{14}$$

The demand curve $B_t^b(p_t)$ is integrated buy order distribution from p_t to the highest price and then they obtained,

$$B_t^b(p_t) = A_t^b - \frac{A_t^b}{2}\left[1 + \mathrm{erf}\left(\frac{p_t - p_{t-1}}{\sqrt{2\acute{\rho}^2}}\right)\right]. \tag{15}$$

As the left side of Fig. 2 shows, in the case of $A_t^s = A_t^b$ the market price p_t is determined as p_{t-1}, and the market price is not changed. On the other hand, as the right side of Fig. 2 shows, in the case of $A_t^s \neq A_t^b$ the market price is not p_t. In short, due to the imbalance of buy-sell orders

*6 Note that in Fig.2 the horizontal axis shows prices and the vertical axis shows cumulative amount of orders.

the market price determined as a different price from the average of investors' estimated price(p_{t-1}).

Here, from $B_t^s(p_t) = B_t^b(p_t)$ they obtained p_t

$$p_t - p_{t-1} = r_t = \sqrt{2\acute{\rho}^2}\mathrm{erf}^{-1}(Z). \quad (16)$$

Where r_t is log return, erf^{-1} is the inverse function of the error function and

$$Z \equiv \frac{A_t^b - A_t^s}{A_t^b + A_t^s}, \quad (17)$$

where Z is always satisfied,

$$|Z| < 1. \quad (18)$$

Here they assumed $Z \ll 1$ and then they could Maclaurin expand $\mathrm{erf}^{-1}(z)$ by the first order term (there is no second order term),

$$r_t = \sqrt{\frac{\pi}{2}}\acute{\rho}Z. \quad (19)$$

Furthermore, they defined $\rho = \sqrt{\pi/2}\acute{\rho}$ and restored Z, finally they obtained,

$$r_t = \rho\frac{A_t^b - A_t^s}{A_t^b + A_t^s}. \quad (20)$$

This equation is exactly Eq.(2).

Disclaimer

Note that the opinions contained herein are solely those of the authors and do not necessarily reflect those of SPARX Asset Management Co., Ltd.

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